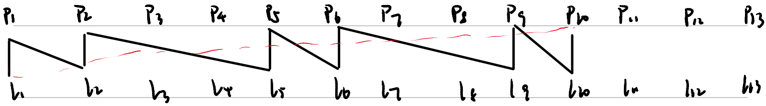
$\mathbb{P}^2\mathbb{F}_3$: points: $P_1, \dots, P_{13}$

lines: $l_1, \dots, l_{13}$

l_j connected to $P_j, P_{j+1}, P_{j+3}, P_{j+9}$

$(3^3-1)/2 = 27/2 = 13.$

A line is a plane consisting of $\frac{9-1}{2} = 4$ points.



Calculate rank: 26 vertices, need to find rank of the radical.

$\langle P_i, P_i \rangle = 3$, fix edges from points to lines.

$\langle P_i, l_j \rangle = \theta \dots$

$\langle n, P_i \rangle = \langle n, l_j \rangle = 0$ for all i, j.

Let $n = a^i P_i + b^j l_j$. $\langle n, P_k \rangle = \langle a^i P_i + b^j l_j, P_k \rangle = 3a^k + b^k \bar{\theta} + b^{k-1}\bar{\theta} + b^{k-3}\bar{\theta} + b^{k-9}\bar{\theta} = 0.$

$\langle n, l_l \rangle = 3b^l + a^k\theta + a^{k+1}\theta + a^{k+3}\theta + a^{k+9}\theta = 0.$

$\theta = \sqrt{-3}$, $\bar{\theta} = -\sqrt{-3}$

$\Rightarrow 3a^k - \sqrt{-3}(b^k + b^{k-1} + b^{k-3} + b^{k-9}) = 0$

$\Rightarrow \sqrt{-3}\, a^k + (b^k + b^{k-1} + b^{k-3} + b^{k-9}) = 0.$

$\sqrt{-3}\, b^l - (a^l + a^{l+1} + a^{l+3} + a^{l+9}) = 0.$

26 equations

$3b^l - (b^l + b^{l-1} + b^{l-3} + b^{l-9} + b^{l+1} + b^l + b^{l-2} + b^{l-8} + b^{l+3} + b^{l+2} + b^l + b^{l-6} + b^{l+9} + b^{l+8} + b^{l+6} + b^l) = 0$

$\Rightarrow b^l + b^{l-1} + b^{l-2} + b^{l-3} + b^{l-6} + b^{l-8} + b^{l-9} + b^{l-12} + b^{l-11} + b^{l-10} + b^{l-7} + b^{l-5} + b^{l-4} = 0.$

Then: radical: $a^l = \cdots b^{l+} \cdots$, and $\sum_{i=1}^{13} b^i = 0.$

$\Rightarrow$ rank of radical is $13 - 1 = 12$. 为什么刚好是完全剩余系？

$L^* = \{x \in L \otimes \mathbb{Q} : \langle x | l \rangle \in \mathbb{Z}$ for all $l \in L\}.$

$L \subseteq \theta L^*$. $\Leftrightarrow \frac{L}{\theta} \subseteq L^* \Leftrightarrow \langle \frac{l}{\theta}, l' \rangle \in \mathbb{Z}$. $\langle \frac{l}{\theta}, l' \rangle \in \mathbb{Z}$ since

$\langle l, l' \rangle$ is divisible by θ in $\mathbb{Z}$.

To see that $\theta L^* \subseteq L$. we need to see that $L^* \subseteq \frac{L}{\theta}$.

For any $l' \in L^*$, let $l' = a^i p_i + b^i l_j$.

Since $\mathrm{rad} = \{ a^k p_k + b^l p_l \mid \sum_l b^l = 0,\ a^k = -\frac{1}{\sqrt{3}}(b^k + b^{k-1} + b^{k+3} + b^{k-9}) \}$

$\Rightarrow L$ can be generated by $p_0, \cdots, p_{13}, (l_0 + \cdots + l_{13})$.

Let T be symmetry group of L, then $\overset{L_3(3)}{\overset{\vee}{PGL_2(\mathbb{F}_3)}} \subseteq T$. since

PGL is a symmetry of lines and points since there is an action

which switches lines and points. we have $L_3(3) \rtimes \mathbb{Z}_2 \subseteq T$.

$$p_i = (0, \cdots, \underset{\substack{\uparrow \\ 0,1,\cdots\ j\ \cdots 13}}{\theta}, \cdots, 0), \qquad l_i = (1, 0, \cdots, 0) + e_i + e_{i+1} + e_{i+3} + e_{i+9}$$

$p_\infty = (\sqrt{3}; 0, \cdots, 0) \qquad l_\infty = (4; 1, \cdots, 1)$

$\langle p_i, p_j \rangle = 0, \quad \langle l_i, l_j \rangle = 0.$

$\langle x, y \rangle = -x_0 \bar{y}_0 + x_1 \bar{y}_1 + \cdots + x_{13} \bar{y}_{13}$

Let $\tau = l_\infty + i p_\infty = (4 + \sqrt{3} i; 1^{13})$